1 Order Matters

It is useful to find a formula for a sequence of numbers, as it helps predict other numbers in the sequence, explore mathematical properties, see the growth rate, and find relationships between different sequences.

Essentially, a sequence is everything that has an order.

The words in a dictionary, different geometric figures drawn one after another, your clothes, hanging in the closet, a list of numbers - they all have an order. The sequences are everywhere, but what do all of them have in common? Every member of the sequence has an index, in order to quickly navigate in the sequence. For example, $a_0, a_1, a_2, a_3, ...a_n$, (n is some countable number), and where 1, 2, 3,...,n is an index. Also, every member of the sequence can be written as a formula - we start with a countable number, do something to it and then get a member of the sequence.

The patterns can be classified!

2 Types of Sequences

2.1 Arithmetic

Arithmetic sequence starts with a number a_1, and then you add a constant number repeatedly. You want to buy several bottles of soda, one bottle costs 1 dollar, but any other one you buy costs two dollars. 1, 3, 5, 7, 9,... - you start with one dollar and add two every time afterwards. Consequently, the formula is $2n - 1$. Identify the following

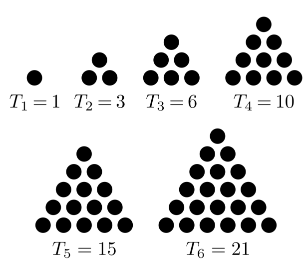
1. Consider the sequence $45, 50, 55, 60, \ldots$ - what is the formula for this?
2. You want to buy ten packages of cheese, and you know that first one costs 5 dollars. Every other one costs 6 dollars. What is the total price of the ten packages? What is the price if you want to buy a) one, b) two, c) three packages instead?

2.2 Geometric

Geometric sequence starts with a number a_1, and then you multiply by a constant number repeatedly. Imagine you have two friends at your party, and each of them may invite two friends as well, and so on and so forth. You start with two friends, than have four more friends, then eight more friends, etc: 2, 4, 8, 16, ... The formula is 2^n.

2.3 Figurate

Figurate sequence is the one that demonstrates the amount of points needed to draw some geometric figures. $1, 3, 6, 10, \ldots$ is a triangular sequence, shown on the picture below. $1, 4, 9, 16, \ldots$ is a square sequence. What is a formula for the square sequence? $1, 6, 15, 28, \ldots$ is a hexagonal sequence.



2.4 Fibonacci

It was introduced to the West by an Italian mathematician Leonardo Fibonacci, who wrote of his findings in the book Liber Abaci. It is a series of numbers; each term is found by adding the two numbers before it, starting with 1 and 1, going up from there.

n	0	1	2	3	4	5	6
F_n	1	1	2	3	5	8	13

Fibonacci can also be written as $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$

2.5 Lucas

It is an integer sequence named after Francois Edouard Anatole Lucas. It is closely related to the Fibonacci sequence in terms of the recursive form of the equation, but starts with 2 and 1 instead of 1 and 1.

n	0	1	2	3	4	5	6
F_n	2	1	3	4	7	11	18

3 Phi

The golden ratio, or ϕ is a crazy constant of mathematics that can be found all over the world. One of the forms of ϕ is the solution to the equation $x^2 - x - 1 = 0$ or approximately 1.618 but it can be seen in many other places. For example, try dividing consecutive terms of the Fibonacci sequence with each other.

F_2/F_1	2
F_3/F_2	1.5
F_4/F_3	1.667
F_5/F_4	1.6
F_6/F_5	1.625
F_7/F_6	1.619

Practice Problems:

1. Do you notice anything special about these values?
2. Up until now, we have defined the Fibonacci numbers recursively, using previous terms to calculate the next term. Can we define F_n as a function of n given the pattern we see?

Another cool thing happens when we raise ϕ to different powers. Let us try this, rounding ϕ^n to the nearest whole number

n	ϕ^n
1	2
2	3
3	4
4	7
5	11

Do we see anything familiar about these numbers?

4 Challenge Problems

1. Bubba wants a formula for this sequence: $-4, 12, -36, \ldots$ Can you provide one? What type of sequence is this?

2. Approximate the value of $(\frac{13}{5})^5$

3. Santa is playing a game with his elves. Each row and each column in the 5×5 array is an arithmetic sequence with five terms. What is the value of X? (AMC8)

1				25
		X		
17				81

4. Donkey the Cow moves along a number line (please do not ask me why). He starts at 0, slides forward 1 unit, slides backwards 2, forward 3, backward 4, and so on. Where will Donkey be after 2011 slides? (Modified from UKMT)

5. Billy wants to construct an $N \times 1$ grid out of toothpicks. If each toothpick covers 1 unit, how many toothpicks does he need to construct the figure if $N = 1$? What if $N = 3$? What if $N = 6$?

6. Billy still has spare toothpicks from the last project. Now he wants to construct an $N \times N$ grid out of toothpicks. If each toothpick covers 1 unit, how many toothpicks does he need to construct the figure if $N = 1$? What if $N = 3$? What if $N = 6$?

7. The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence. (2012 AIME)